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On Position-Space Renormalization Group Approach to Percolation

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In a position-space renormalization group (PSRG) approach to percolation one calculates the probability $R(p, b)$ that a finite lattice of linear size b percolates, where p is the occupation probability of a site or bond. A sequence of percolation thresholds $p_c(b)$ is then estimated from $R(p_c, b) = p_c(b)$ and extrapolated to the limit $b \to \infty$ to obtain $p_c = p_c(\infty)$. Recently, it was shown that for a certain spanning rule and boundary condition, $R(p_c, \infty) = R_c$ is universal, and since p_c is not universal, the validity of PSRG approaches was questioned. We suggest that the equation $R(p_c, b) = \alpha$, where α is *any* number in (0, 1), provides a sequence of $p_c(b)$'s that *always* converges to p_c as $b \to \infty$. Thus, there is an *envelope* from any point inside of which one can converge to p_c. However, the convergence is optimal if $\alpha = R_c$. By calculating the fractal dimension of the sample-spanning cluster at p_c, we show that the same is true about *any* critical exponent of percolation that is calculated by a PSRG method. Thus PSRG methods are still a useful tool for investigating percolation properties of disordered systems.

KEY WORDS: Percolation; renormalization; universality.

1. INTRODUCTION

Percolation theory has become a powerful, much-used tool for investigating various phenomena in disordered media.[1] Its popularity stems from its relevance to a wide variety of fields[2] and from the fact that despite the simplicity of its underlying concepts, it leads to nontrivial critical phenomena. Moreover, much like the Ising model of statistical mechanics, percolation has become a testing ground for various computational

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approaches to critical phenomena. Despite extensive studies, few exact results have been obtained for the most interesting quantities in percolation theory, and in particular the percolation thresholds p_c and various critical exponents that characterize the nonanalytical behavior of percolation quantities near p_c. Thus, over the years many approaches have been developed for estimating such properties. For nearly two decades position-space renormalization group (PSRG) methods have been an accurate tool for calculating the percolation threshold and the critical exponents of percolation. However, a recent paper by Ziff[3] casts doubts on the validity or usefulness of this approach. The purpose of this paper is to further study this issue and to suggest that PSRG methods are still a useful tool for studying percolation. We first describe briefly the PSRG approach and the results of Ziff[3] and those of others[4-6] relevant to this problem, and then present our own calculations and results.

2. POSITION-SPACE RENORMALIZATION GROUP APPROACH

The essence of a PSRG approach[7-11] to percolation is as follows. Consider, for example, a square lattice in which each site is occupied with probability p and is empty with probability $1-p$. We partition the lattice into $b \times b$ cells, where b is the number of sites in each direction. We then replace each cell with a single site whose probability of being occupied is $R(p, b)$, called the RG transformation, and in effect is the probability that a cell of linear size b percolates. To calculate $R(p, b)$ one specifies a spanning rule and finds *all* RG cell configurations, with some sites occupied and some empty, that percolate according to the spanning rule. The RG transformation is then the *sum* of the probabilities of all percolating configurations in the cell. Various spanning rules have been used in the past (for reviews see Payandeh,[9] Stanley *et al.*,[10] and Family[11]). Reynolds *et al.*[12] proposed three basic spanning rules in two dimensions (2D), which are R_0, which considers a cell as occupied if a cluster of occupied sites spans the cell *either* horizontally or vertically, R_1, according to which a cell is occupied if the cluster spans it in a *fixed* direction, and R_2, which is the rule if the cluster spans the cell *both* horizontally and vertically. For example, for a 3×3 RG cell rule, R_0 yields[12]

$$R(p, b) = p^9 + 9p^8q + 36p^7q^2 + 82p^6q^3 + 93p^5q^4 + 44p^4q^5 + 6p^3q^6 \quad (1)$$

where $q = 1 - p$, while rule R_1 gives[12]

$$R(p, b) = p^9 + 9p^8q + 36p^7q^2 + 67p^6q^3 + 59p^5q^4 + 22p^4q^5 + 3p^3q^6 \quad (2)$$

and rule R_2 results in[12]

$$R(p, b) = p^9 + 9p^8q + 36p^7q^2 + 52p^6q^3 + 25p^5q^4 \tag{3}$$

At p_c, the sample-spanning cluster is self-similar and fractal. Thus, p_c is an invariant of the transformation and may be estimated from

$$R(p_c, b) = p_c(b) \tag{4}$$

where $p_c(b)$ is an approximate estimate of p_c that one obtains with a RG cell of linear size b. Thus, if one calculates $R(p, b)$ for a sequence of b's, one obtains a sequence of $p_c(b)$'s which can then be extrapolated to $b \to \infty$ to obtain p_c. Finite-size scaling predicts that

$$p_c(b) - p_c \sim b^{-1/\nu} \tag{5}$$

where ν is the critical exponent that characterizes the divergence of the correlation length ξ_p as p_c is approached, $\xi_p \sim (p - p_c)^{-\nu}$. For site percolation on the square lattice, $p_c \simeq 0.59275$, and for 2D percolation, $\nu = 4/3$.

However, Ziff[3] recently showed that on the square lattice rule R_1 does *not* renormalize at p_c. Specifically, he showed that as $b \to \infty$, $R(p_c, b \to \infty) = R_c$ approaches a *constant* value

$$R_c \to \tfrac{1}{2} \tag{6}$$

Since Bernasconi[13] had already shown that for bond percolation on the square lattice $R(p_c, b) = 1/2$ for *any* b, Eq. (6) implies that R_c is *universal*, in contradiction with Eq. (4), since p_c is not universal, and Eq. (4) is supposed to be valid for any b. Moreover, Ziff[3] showed that on the square lattice

$$p_c(b) - p_c \sim b^{-1-1/\nu} \tag{7}$$

which implies a *faster* convergence of $p_c(b)$ to p_c than indicated by Eq. (5). Ziff's results were generalized by Aharony and Hovi,[4] who also suggested that the critical exponent in Eq. (7) might be somewhat smaller than $1 + 1/\nu = 7/4$ in 2D. Stauffer *et al.*[5] showed that for site and bond percolation on the simple-cubic network, $R_c \simeq 0.42$ is indeed universal, but that in 3D, Eq. (5) is still valid. Gropengiesser and Stauffer[6] showed that the universality of R_c is somewhat weak, since it may depend on the boundary conditions used in the lattice. Ziff[3] stated that these results indicate that the application of PSRG methods to percolation, at least with rule R_1, is problematic, casting doubt on their validity and usefulness for estimating p_c and other important properties. The purpose of this paper is to show that, although for rule R_1, R_c is universal, PSRG methods can still be used for estimating p_c and other critical properties of percolation networks.

3. CALCULATIONS AND DISCUSSION

We propose that if one solves

$$R(p_c, b) = \alpha \tag{8}$$

where α is an *arbitrary* number in (0, 1), then this equation results, with increasing b, in a sequence of $p_c(b)$'s which converges to p_c. There are, however, two important points to remember. One is that although α can be *any* number in (0, 1), and the resulting sequence of $p_c(b)$'s always converges to p_c, the convergence is optimal if $\alpha = R_c$, the universal value of $R(p_c, \infty)$. For example, on the square lattice if we choose $\alpha = 1/2$ for PSRG transformations obtained with rule R_1, we obtain very fast convergence, as indicated by Eq. (7). The second important point is that equations $R(p_c, b) = 0$ and $R(p_c, b) = 1$ do *not*, in general, provide any estimates of $p_c(b)$, since the former equation corresponds to an empty lattice (the lattice animal limit), while the latter equation corresponds to a completely occupied lattice. Thus, if we set $\alpha = \varepsilon$ and $\alpha = 1 - \varepsilon$, where ε is a small but nonzero number, we obtain the lower and upper limits of the sequence of $p_c(b)$'s, and therefore we have an *envelope* from any point inside of which one can start a path toward p_c. Of course, we cannot take α to be negative or larger than 1, as $R(p, b)$ is a probability.

To prove these, we used the RG transformations for rules R_0, R_1, and R_2 on the square lattice, set $\alpha = 0.01$, 1/2, and 0.99 (i.e., $\varepsilon = 0.01$), and solved the resulting equations to estimate $p_c(b)$. For rule R_0 the RG transformations were derived exactly by Reynolds *et al.*[12] for up to $b = 5$ cells, while for rules R_1 and R_2 they were derived for up to $b = 4$ cells. Ziff[3] extended the R_1 results to $b = 7$, and we extend the R_2 results to $b = 5$. Note that Ziff[3] and others[4-6] discussed only the R_1 rule, but here we also discuss the other two rules.

Figure 1 shows the results for rule R_0. As can be seen, the results for $\alpha = 1/2$ converge smoothly to p_c if they are plotted as $p_c(b)$ versus $(b + 5.5)^{-1.75}$. The exponent 1.75 is nothing but $1 + 1/\nu$, as suggested by Ziff,[3] our Eq. (7). The constant 5.5 is necessary for taking into account the effect of correction-to-scaling terms, since Eq. (7) is applicable to large values of b, and for small and moderate b one has to take into account such correction terms. Figure 1 shows that the results with $\alpha = 0.01$ and 0.99 also converge to p_c, although the convergence is not similar to Eq. (7), and thus these three curves define the envelope discussed above. Thus if we take α to be any number in (0, 1) and solve Eq. (8), we obtain a curve inside the envelope that with increasing b will pass through p_c. The fact that the results for $\alpha = 1/2$ fall on the straight line implies that the asymptotic value of R_c for rule R_0 is either 1/2 or very close to it. Note also that the results

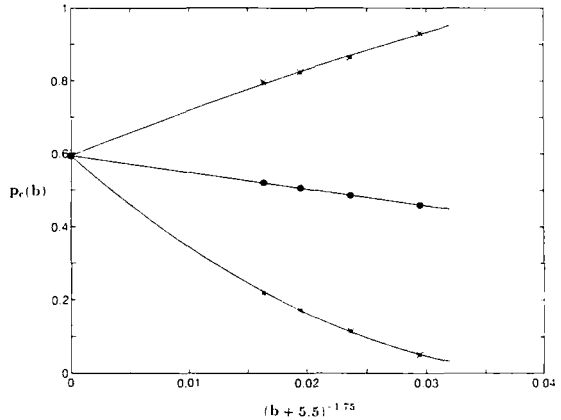

Fig. 1. Dependence of $p_c(b)$ on b for rule R_0 on the square lattice. The curves are, from top to bottom, for $\alpha = 0.99$, 1/2, and 0.01.

for $\alpha = 0.99$ almost are on a straight line, implying that for this value of α the convergence to p_c is close to what is suggested by Eq. (7), but perhaps with an exponent somewhat smaller than $1 + 1/\nu$ but larger than $1/\nu$, although our results are not accurate enough to distinguish between $1 + 1/\nu = 7/4$ (in 2D) and a somewhat smaller exponent. On the other hand, the results for $\alpha = 0.01$ converge more slowly to p_c. Thus, the convergence of $p_c(b)$'s to p_c is optimal when $\alpha = R_c$.

Figure 2 shows that results for rule R_1. The results for $\alpha = 1/2$ were also obtained by Ziff.[3] Note, however, that the straight line for $\alpha = 1/2$ is obtained if p_c is plotted versus $(b + 1.4)^{-1.75}$, somewhat different from Fig. 1.

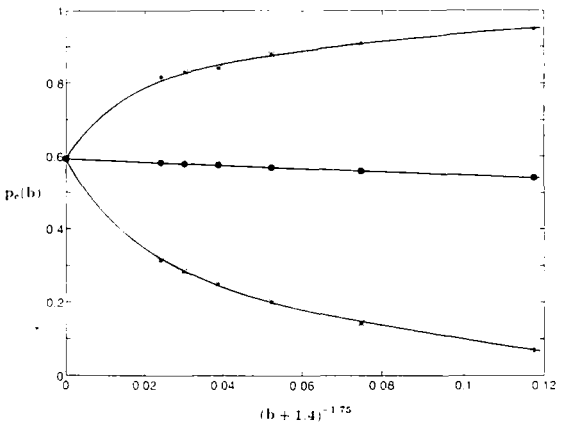

Fig. 2. Same as in Fig. 1, but for rule R_1.

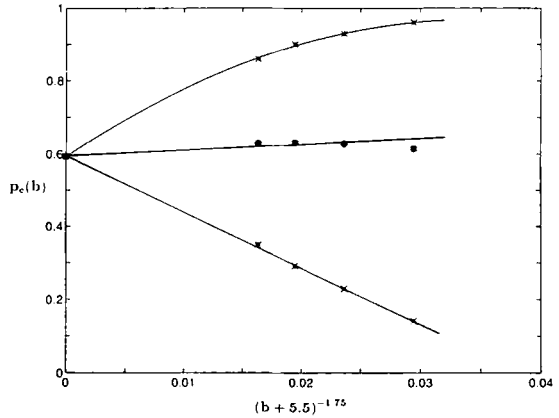

Fig. 3. Same as in Fig. 1, but for rule R_2.

Since the constants 5.5 and 1.4 represent only the correction-to-scaling terms, these results confirm our assertion that for rule R_0 the value of R_c is either 1/2 or very close to it. Note that the results for this case clearly indicate that the convergence to p_c for any α other than 1/2 is slower than what is suggested by Eq. (7), confirming our assertion. Figure 3 shows the results for rule R_2. In this case the sequence of $p_c(b)$'s for $\alpha = 1/2$ does not converge to $p_c \simeq 0.59275$ as fast as those for rules R_0 and R_1, indicating that for this rule R_c is *not* 1/2. This can be attributed to the fact that the boundary conditions for R_2 (which requires spanning in *both* directions) are not the same as those of R_1 (spanning in *one fixed* direction), and Gropengiesser and Stauffer[6] already showed that R_c may depend on the boundary conditions.

To further confirm these results, we also calculated the fractal dimension d_f of the sample-spanning cluster at p_c, using a method developed by Family and Reynolds.[14] In this method, a weight or fugacity K is associated with every occupied site of the RG cell, and a weight $q = 1 - p$ is assigned to every empty site. The fugacity of the renormalized site is K'. Thus, for example, Eq. (1) becomes

$$K' = K^9 + 9K^8q + 36K^7q^2 + 82K^6q^3 + 93K^5q^4 + 44K^4q^5 + 6K^3q^6 \quad (9)$$

A fixed point $K_c(b)$ is then calculated from $K'(K_c, q_c, b) = K_c$, where $q_c(b) = 1 - p_c(b)$ as calculated above. An estimate of d_f is given by

$$d_f(b) = \frac{\ln \lambda}{\ln b} \quad (10)$$

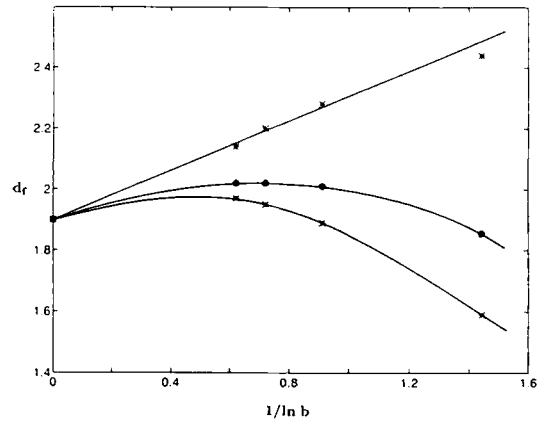

Fig. 4. Dependence of the fractal dimension $d_f(b)$ on b for rule R_0 on the square lattice.

where $\lambda = dK'/dK$, evaluated at $K_c(b)$ and $q_c(b)$. We used this method and the RG transformations for rules R_0, R_1, and R_2. Three sets of $q_c(b)$'s were also used which correspond to those obtained above for $\alpha = 0.01$, 1/2, and 0.99. The results for rule R_0 are shown in Fig. 4 (with similar results for R_1 and R_2, which are not shown), where we plot $d_f(b)$ versus $1/\ln(b)$ (which is the standard extrapolation method for the critical exponents). As can be seen, the results in all cases appear to converge to $d_f = 91/48 \simeq 1.9$ for 2D percolation. Thus, using the RG transformations, one can obtain an envelope for the critical exponents of percolation, from any point inside of which it is possible to converge to the critical exponents of the infinite system.

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